Wilson's Twistors: Non-Toroidal Maximal Abelian Subgroups for Finite-Group Quantisation of SL(4,R)

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Abstract

The real form $SL(4,\mathbb{R})$ of the twistor group $SL(4,\mathbb{C})$ admits a finite-group quantisation at the electroweak stage, recently proposed by R. A. Wilson. His construction uses two quaternion groups $Q_8^{(L)}$ and $Q_8^{(R)}$ inside $SU(2)_L \times SU(2)_R$, together with distinct left- and right-handed triplet extensions, producing a finite subgroup of order 288 with normal core $Q_8^{(L)} \times Q_8^{(R)}$ and quotient $S_3 \times \mathbb{Z}_3$. This group encodes parity violation, flavour directionality, mass hierarchy, and the observed asymmetry between $SU(2)_L$ and $SU(2)_R$.

We construct Wilson's finite-group quantisation using the modern perspective of non-toroidal maximal abelian subgroups (MASAs) of Lie groups. Flavour symmetry is shown to be intrinsically non-toroidal, colour symmetry toroidal, and weak symmetry a mixture of the two. These structural facts force the left-handed $SU(2)$ to contain a non-toroidal MASA and the right-handed $SU(2)$ to contain a toroidal MASA. We prove that Wilson's group H_{288} is the unique minimal finite subgroup of $SU(2)_L \times SU(2)_R$ satisfying these constraints.

We embed H_{288} into the full twistor chain $SU(2,2) \subset SL(4,\mathbb{C}) \supset SL(4,\mathbb{R}) \supset SO(3,3) \supset SO(4)$, and show how it provides a discrete twistor-like structure for the $SL(4,\mathbb{R})$ branch. Connections are made with the continuous vacuum-polarisation framework based on the compact real form $SU(4)$, the emergent colour group $SU(3)_{\text{eff}}$ from $SL(3,\mathbb{R})$, and the Euclidean quantum-field-theoretic derivation of Lorentz symmetry and the corrected chiral Dirac equation.

The result is a unified mathematical and physical picture in which Wilson's finite group H_{288} appears as the discrete MASA shadow of the $SL(4,\mathbb{R})$ electroweak real form, uniquely encoding the non-toroidal flavour geometry and the toroidal colour geometry required by twistor-based unification.

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1 Introduction

The twistor group $SL(4,\mathbb{C}) = \mathrm{Spin}(6,\mathbb{C})$ admits three real forms relevant to four-dimensional physics:

$$SU(4), \qquad SL(4,\mathbb{R}), \qquad SU(2,2).$$

The compact form $SU(4)$ underlies vacuum-polarisation dynamics and the emergent Lorentzian geometry of quantum fields. The conformal form $SU(2,2)$ is the classical Penrose twistor group. The split real form $SL(4,\mathbb{R})$

provides an alternative twistor branch in which electroweak symmetry appears naturally as

$$SL(4,\mathbb{R}) \supset SO(3,3) \supset SO(4) \simeq (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2.$$

Wilson has proposed that this $SL(4,\mathbb{R})$ electroweak stage should be quantised by a finite subgroup of $SO(4)$ built from two quaternion groups $Q_8^{(L)}$ and $Q_8^{(R)}$, together with distinct left- and right-handed triplet extensions. His construction yields a group of order 288, denoted H_{288}, with normal core $Q_8^{(L)} \times Q_8^{(R)}$ and quotient $S_3 \times \mathbb{Z}_3$. This finite group encodes the observed chirality of the weak interaction, the three-generation flavour structure, and the left-right asymmetry of electroweak physics.

Wilson's finite-group quantisation is based on maximal abelian subgroups (MASAs) of Lie groups. Toroidal MASAs (Cartan subgroups) are familiar in physics, but non-toroidal MASAs are rare and have been largely ignored. Wilson's recent observation that flavour symmetry is non-toroidal, colour symmetry toroidal, and weak symmetry mixed provides the missing conceptual framework for understanding why his finite group H_{288} is not merely plausible but essentially forced.

We show that the left-handed $SU(2)$ must contain a non-toroidal MASA, the right-handed $SU(2)$ must contain a toroidal MASA, and that the only finite subgroup of $SU(2)_L \times SU(2)_R$ satisfying these constraints is Wilson's group H_{288}. We then embed this finite group into the full twistor chain and relate it to the continuous structures arising from $SU(4)$ vacuum polarisation, the emergent colour group $SU(3)_{\text{eff}}$ from $SL(3,\mathbb{R})$, and the Euclidean quantum-field-theoretic derivation of Lorentz symmetry and the corrected chiral Dirac equation.

The result is a unified mathematical and physical picture in which Wilson's finite group H_{288} appears as the discrete twistor fibre of the $SL(4,\mathbb{R})$ branch, uniquely encoding the non-toroidal flavour geometry and the toroidal colour geometry required by twistor-based unification.

2 Twistor Geometry and the Real Forms of $SL(4,\mathbb{C})$

The complex group

$$SL(4,\mathbb{C}) = \mathrm{Spin}(6,\mathbb{C})$$

is the double cover of the complexified six-dimensional rotation group. It is the natural twistor group for four-dimensional physics, since its fundamental representation $\mathbb{C}^4$ is one of the two half-spinor representations of $\mathrm{Spin}(6,\mathbb{C})$.

The real forms of $SL(4,\mathbb{C})$ relevant to physics are:

$$SU(4), \qquad SL(4,\mathbb{R}), \qquad SU(2,2).$$

Each real form corresponds to a distinct physical interpretation:

- The compact real form $SU(4)$ underlies vacuum-polarisation dynamics, emergent Lorentz symmetry, and the corrected chiral Dirac equation. Its Lie algebra $\mathfrak{su}(4)$ is isomorphic to $\mathfrak{so}(6)$.
- The split real form $SL(4,\mathbb{R})$ underlies Wilson's finite-group quantisation of the electroweak sector. Its Lie algebra $\mathfrak{sl}(4,\mathbb{R})$ is isomorphic to $\mathfrak{so}(3,3)$.
- The non-compact real form $SU(2,2)$ is the conformal group of four-dimensional Minkowski space and the classical Penrose twistor group.

These three real forms fit into the twistor chain:

$$SU(2,2) \ \subset \ SL(4,\mathbb{C}) \ \supset \ SL(4,\mathbb{R}),\, SU(4).$$

The split real form $SL(4,\mathbb{R})$ contains the electroweak structure

$$SL(4,\mathbb{R}) \supset SO(3,3) \supset SO(4) \simeq (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2,$$

and it is precisely at this $SO(4)$ stage that Wilson's finite-group quantisation appears.

The compact real form $SU(4)$ contains the decomposition

$$SU(4) \supset SU(3) \times U(1)_G \supset SU(2) \times U(1)_Y,$$

which underlies the continuous vacuum-polarisation interpretation of electroweak and strong interactions.

The conformal real form $SU(2,2)$ contains the twistor description of Minkowski spacetime:

$$\mathbb{CP}^3 \supset \mathrm{Gr}(2,4,\mathbb{C}),$$

with spacetime points represented as projective lines in twistor space.

In this monograph we show that Wilson's finite group H_{288} is the *discrete twistor fibre* of the $SL(4,\mathbb{R})$ branch, in precise analogy with the continuous twistor fibres of the $SU(2,2)$ and $SU(4)$ branches. The key mathematical tool enabling this interpretation is the theory of maximal abelian subgroups (MASAs), both toroidal and non-toroidal.

3 Toroidal and Non-Toroidal Maximal Abelian Subgroups

A maximal abelian subgroup (MASA) of a Lie group G is a maximal subgroup $A \subset G$ such that every pair of elements of A commute. MASAs come in two fundamentally different types:

- **Toroidal MASAs**: Cartan subgroups, consisting of diagonal matrices or commuting rotations. These are familiar in physics:

$$U(1)^n \subset SU(n), \qquad \text{diagonal generators.}$$

- **Non-toroidal MASAs**: abelian subgroups generated by non-commuting matrices which nevertheless commute within the MASA. These are rare and structurally rigid.

Examples of non-toroidal MASAs include:

Pauli matrices $SO(3)$, Dirac matrices $SO(6)$, Gell-Mann matrices $PSU(3)$.

These MASAs are not diagonal and do not lie in any Cartan torus.
Wilson has emphasised that:

- **Flavour symmetry is non-toroidal**. It cannot be expressed in terms of diagonal matrices or Weyl-group reflections. This explains why three-generation flavour physics has resisted conventional Lie-group classification.

- **Colour symmetry is toroidal**. It lies in a Cartan torus of $SU(2)$ rather than the Weyl group of $SU(3)$, consistent with the emergent colour interpretation $SL(3,\mathbb{R}) \to SU(3)_{\text{eff}}$.

- **Weak symmetry is mixed**. The left-handed $SU(2)_L$ contains a non-toroidal MASA, while the right-handed $SU(2)_R$ contains a toroidal MASA.

These structural facts force the following finite-group behaviour:

- The left-handed $SU(2)_L$ must contain a finite non-toroidal MASA, which is realised by the semidirect product $\mathbb{Z}_3 \rtimes Q_8$.

- The right-handed $SU(2)_R$ must contain a finite toroidal MASA, realised by the semidirect product $Q_8 \rtimes \mathbb{Z}_3$.

- The combined finite subgroup of $SU(2)_L \times SU(2)_R$ must contain both structures simultaneously.

We show in later sections that the only finite subgroup satisfying these requirements is Wilson's group H_{288}, with normal core $Q_8^{(L)} \times Q_8^{(R)}$ and quotient $S_3 \times \mathbb{Z}_3$.

Thus the MASA classification provides the conceptual foundation for Wilson's finite-group quantisation of $SL(4,\mathbb{R})$.

4 The Quaternionic Model of $SO(4)$

The group $SO(4)$ admits the well-known double-cover decomposition

$$SO(4) \;\simeq\; (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2,$$

where the two $SU(2)$ factors act on the quaternion algebra

$$\mathbb{H} = \{\, x_0 + x_1 i + x_2 j + x_3 k \,\} \simeq \mathbb{R}^4$$

by left and right multiplication.
Let $q_L, q_R \in SU(2) \subset \mathbb{H}$ be unit quaternions. Their action on $x \in \mathbb{H}$ is

$$(q_L, q_R) :\; x \mapsto q_L\, x\, q_R^{-1}.$$

This defines a homomorphism

$$SU(2)_L \times SU(2)_R \longrightarrow SO(4),$$

whose kernel is $\{(-1,-1)\}$, giving the quotient by $\mathbb{Z}_2$.

In the basis $\{1, i, j, k\}$, left multiplication by i is represented by the matrix

$$L_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and similarly for L_j, L_k and the right actions R_i, R_j, R_k.

Thus $SU(2)_L$ and $SU(2)_R$ embed explicitly into $SO(4)$ as two commuting copies of the quaternionic rotation group.

This quaternionic model is the natural setting for Wilson's finite-group quantisation, since the quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

is a finite subgroup of $SU(2)$, and hence of both $SU(2)_L$ and $SU(2)_R$. The direct product

$$Q_8^{(L)} \times Q_8^{(R)} \subset SU(2)_L \times SU(2)_R$$

is therefore a canonical 16-element finite subgroup of $SO(4)$.

This $Q_8^{(L)} \times Q_8^{(R)}$ core is the foundation of Wilson's finite group H_{288}.

5 Finite MASAs Inside $SL(4, \mathbb{R})$

Finite subgroups of $SU(2)$ are completely classified:

$$Q_8, \qquad 2T, \qquad 2O, \qquad 2I,$$

the quaternion, binary tetrahedral, binary octahedral, and binary icosahedral groups. These are the finite subgroups of the three-sphere S^3.

Wilson's finite-group quantisation uses the following structural facts:

- The left-handed $SU(2)_L$ must contain a *non-toroidal* MASA. Finite non-toroidal MASAs correspond to semidirect products $\mathbb{Z}_3 \rtimes Q_8$.

- The right-handed $SU(2)_R$ must contain a *toroidal* MASA. Finite toroidal MASAs correspond to semidirect products $Q_8 \rtimes \mathbb{Z}_3$.
- The combined finite subgroup of $SU(2)_L \times SU(2)_R$ must contain both structures simultaneously.

The relevant 24-element extensions of Q_8 are:

- The binary tetrahedral group $2T$, generated by Q_8 and

$$t = \frac{-1 + i + j + k}{2}.$$

- The dicyclic group Dic_{12}, generated by Q_8 and a $\mathbb{Z}_3$ element acting nontrivially on Q_8.

These two extensions correspond precisely to the two MASA types:

- $2T$ is a finite *toroidal* MASA extension.
- Dic_{12} is a finite *non-toroidal* MASA extension.

Thus the only finite MASAs available inside $SU(2)_L$ and $SU(2)_R$ are:

$$\mathbb{Z}_3 \rtimes Q_8 \quad \text{(non-toroidal)}, \qquad Q_8 \rtimes \mathbb{Z}_3 \quad \text{(toroidal)}.$$

These are the building blocks of Wilson's group H_{288}.

6 Construction of Wilson's Finite Group H_{288}

We now construct Wilson's finite subgroup of $SO(4)$ explicitly.

6.1 Left-handed non-toroidal triplet

Choose the imaginary quaternion i as the distinguished weak-isospin direction. Adjoin to $Q_8^{(L)}$ the element

$$s_L = \frac{-1 + i\sqrt{3}}{2},$$

which generates a $\mathbb{Z}_3$ acting nontrivially on $Q_8^{(L)}$.
This produces the 24-element non-toroidal extension

$$G_L = \langle Q_8^{(L)},\, s_L \rangle \;\subset\; SU(2)_L.$$

6.2 Right-handed toroidal triplet

Adjoin to $Q_8^{(R)}$ the tetrahedral element

$$t_R = \frac{-1+i+j+k}{2},$$

which generates a $\mathbb{Z}_3$ acting trivially on the torus of $SU(2)_R$.
This produces the 24-element toroidal extension

$$2T_R = \langle Q_8^{(R)},\, t_R \rangle \;\subset\; SU(2)_R.$$

6.3 Product and quotient

The direct product

$$G_L \times 2T_R$$

has order $24 \times 24 = 576$.
Quotienting by the central element $(-1,-1)$ gives a subgroup of order

$$|H_{288}| = 576/2 = 288.$$

Thus Wilson's finite group is

$$H_{288} = (G_L \times 2T_R)/\mathbb{Z}_2 \;\subset\; SU(2)_L \times SU(2)_R \subset SO(4).$$

6.4 Structure

The group H_{288} has:

- A normal core

$$Q_8^{(L)} \times Q_8^{(R)},$$

  of order 16.

- A quotient

$$H_{288}/(Q_8^{(L)} \times Q_8^{(R)}) \simeq S_3 \times \mathbb{Z}_3,$$

  of order 18.

This quotient encodes:

- S_3: flavour permutations (discrete CKM-like mixing),
- $\mathbb{Z}_3$: flavour phases (mass hierarchy directionality).

Thus H_{288} is the unique finite subgroup of $SU(2)_L \times SU(2)_R$ containing:

- a non-toroidal MASA on the left,
- a toroidal MASA on the right,
- the quaternionic core $Q_8^{(L)} \times Q_8^{(R)}$,
- and the triplet quotient $S_3 \times \mathbb{Z}_3$.

7 Uniqueness of Wilson's Finite Group H_{288}

Wilson's construction of H_{288} is not merely a clever choice of finite subgroups. Under physically motivated constraints, it is essentially the *unique* finite subgroup of $SU(2)_L \times SU(2)_R$ capable of encoding the observed electroweak and flavour structure.

We impose the following constraints:

(C1) **Finite subgroups of $SU(2)$ only.** The only non-abelian finite subgroups of $SU(2)$ are Q_8, $2T$, $2O$, $2I$.

(C2) **Presence of Q_8 and $\mathbb{Z}_3$.** Discrete spin (Q_8) and three flavours ($\mathbb{Z}_3$) must both appear.

(C3) **Chirality.** $SU(2)_L$ and $SU(2)_R$ must have *different* finite subgroups.

(C4) **Parity violation.** The left-handed subgroup must break a symmetry preserved by the right-handed subgroup.

(C5) **Flavour directionality and mass hierarchy.** The finite group must distinguish directions in flavour space.

(C6) **Clean embedding into $SO(4) \subset SL(4, \mathbb{R})$.** The finite subgroup must sit naturally inside the quaternionic model of $SO(4)$.

Under these constraints, the possible 24-element extensions of Q_8 are:

- $\mathbb{Z}_3 \rtimes Q_8$ (non-toroidal),
- $Q_8 \rtimes \mathbb{Z}_3$ (toroidal),
- $2T$ (toroidal),
- $2O$ (too large),
- $2I$ (too large).

The symmetric choices

$$G_L = G_R$$

violate (C3) and (C4): no chirality, no parity violation.

The direct product choices

$$\mathbb{Z}_3 \times Q_8$$

violate (C4) and (C5): no flavour directionality, no mass hierarchy.

The large binary groups $2O$ and $2I$ violate (C5) and (C6): too many symmetries, no minimal mechanism for chirality or flavour directionality.

Thus the only viable asymmetric choice is:

$$SU(2)_L : \ \mathbb{Z}_3 \rtimes Q_8, \qquad SU(2)_R : \ Q_8 \rtimes \mathbb{Z}_3.$$

The product of these two 24-element groups, modulo the central $\mathbb{Z}_2$, is precisely Wilson's group:

$$H_{288} = (G_L \times 2T_R)/\mathbb{Z}_2.$$

Conclusion. Under constraints (C1)–(C6), Wilson's group H_{288} is the *unique* finite subgroup of $SU(2)_L \times SU(2)_R$ capable of encoding:

- weak chirality,
- parity violation,
- three-generation flavour structure,
- flavour directionality,
- mass hierarchy,
- and clean embedding into $SL(4, \mathbb{R})$.

8 Wilson's Cube–Prism Geometry

Wilson has introduced a geometric model for flavour and colour based on two simple polyhedra:

- a cube (flavour),
- a hexagonal prism (colour).

This geometry provides an intuitive picture of the finite-group structure of H_{288}.

8.1 Flavour: the cube

The cube has 12 edges. Wilson identifies these with the 12 fundamental fermions:

$$\nu_e,\, e,\, u,\, d, \qquad \nu_\mu,\, \mu,\, c,\, s, \qquad \nu_\tau,\, \tau,\, t,\, b.$$

Each edge carries an orientation, distinguishing particle from antiparticle.

There are three distinct projections of the cube onto a square. Each projection selects a set of four fermions:

$$\{\nu,\, e,\, u,\, d\}.$$

These correspond to the three generations.

The flavour triplet $\mathbb{Z}_3$ acts by rotating the cube through 120°, permuting the three projections. This is the geometric origin of the flavour quotient $\mathbb{Z}_3$ in H_{288}.

The permutation group S_3 acts by permuting the three projections. This is the geometric origin of the S_3 quotient in H_{288}.

Thus the flavour quotient

$$S_3 \times \mathbb{Z}_3$$

is realised geometrically by the symmetries of the cube.

8.2 Colour: the hexagonal prism

The hexagonal prism has 6 square faces. Wilson identifies these with the six coloured quarks:

$$u_r,\ u_g,\ u_b, \qquad d_r,\ d_g,\ d_b.$$

Rotation of the prism through 120° permutes the three colours:

$$(r, g, b) \mapsto (g, b, r).$$

This is a toroidal symmetry: the colours lie in a plane and sum to zero.

The hexagonal faces correspond to the leptons, which are invariant under colour rotations.

Thus colour symmetry is toroidal, consistent with the toroidal MASA in $SU(2)_R$.

8.3 Tensor product structure

The full set of fermions is described by the tensor product of the flavour representation (cube) and the colour representation (prism). This tensor product structure is encoded discretely in H_{288}.

Conclusion. Wilson's cube–prism geometry provides a concrete geometric realisation of the quotient $S_3 \times \mathbb{Z}_3$ and the toroidal/non-toroidal MASA structure of H_{288}.

9 Relation to $SL(3,\mathbb{R}) \to SU(3)_{\text{eff}}$

Wilson has proposed that the strong interaction arises not from a fundamental $SU(3)$ gauge group but from the compactification of the split real form

$$SL(3,\mathbb{R}),$$

interpreted as the symmetry of neutrino–antineutrino composite shells.

9.1 Neutrino–antineutrino shells

The Lie algebra

$$\mathfrak{sl}(3,\mathbb{R}) = \{A \in M_3(\mathbb{R}) : \mathrm{tr}(A) = 0\}$$

describes deformations of a three-shell neutrino vacuum. Compactification of this split form yields an effective colour group:

$$SL(3,\mathbb{R}) \longrightarrow SU(3)_{\text{eff}}.$$

In this picture, gluons are composite excitations of the neutrino vacuum:

$$g_a \sim (\nu\bar{\nu})_a, \qquad a = 1, \ldots, 8.$$

9.2 Non-toroidal vs toroidal triplets

The non-commuting triplet symmetries in H_{288} correspond, at the continuous level, to the non-commuting generators of $\mathfrak{sl}(3,\mathbb{R})$.

The toroidal colour symmetry corresponds to the compact Cartan torus of $SU(3)_{\text{eff}}$.

Thus:

- flavour is non-toroidal (split real form),
- colour is toroidal (compact form),
- weak is mixed (split on the left, compact on the right).

9.3 Discrete–continuous correspondence

The finite group H_{288} provides the discrete analogue of the continuous structure:

$$SL(3,\mathbb{R}) \to SU(3)_{\text{eff}}.$$

The non-toroidal MASA in $SU(2)_L$ corresponds to the non-toroidal MASA in $SL(3,\mathbb{R})$.

The toroidal MASA in $SU(2)_R$ corresponds to the toroidal MASA in $SU(3)_{\text{eff}}$.

Conclusion. Wilson's finite group H_{288} encodes, in discrete form, the same toroidal and non-toroidal triplet structure that appears continuously in the strong sector $SL(3,\mathbb{R}) \to SU(3)_{\text{eff}}$.

10 Relation to the Compact Real Form $SU(4)$ and Vacuum Polarisation

The compact real form

$$SU(4) = \text{Spin}(6)$$

provides the continuous twistor branch complementary to the split real form $SL(4,\mathbb{R}) = \text{Spin}(3,3)$. In the $SU(4)$ branch, vacuum polarisation of the $SU(2)$ gauge fields produces:

- emergent Lorentz symmetry,
- the corrected chiral Dirac equation,
- the weak interaction,
- the emergent colour group $SU(3)_{\text{eff}}$,
- and the gravitational/dark-energy $U(1)_G$ field.

The decomposition

$$SU(4) \supset SU(3) \times U(1)_G$$

is the compact analogue of the split decomposition

$$SL(4,\mathbb{R}) \supset SO(3,3) \supset SO(4) \simeq (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2.$$

10.1 Continuous MASAs in $SU(4)$

The compact real form contains both toroidal and non-toroidal MASAs:

- Toroidal MASAs: diagonal generators of $SU(4)$.

- Non-toroidal MASAs: the Dirac matrices, forming a maximal abelian subgroup of $SO(6)$.

Wilson has emphasised that the Dirac matrices form a non-toroidal MASA of $SO(6)$, and that the Gell-Mann matrices form a non-toroidal MASA of $PSU(3)$. These continuous non-toroidal MASAs correspond precisely to the discrete non-toroidal MASAs appearing in H_{288}.

10.2 Discrete–continuous correspondence

The left-handed non-toroidal MASA in H_{288} corresponds to the non-toroidal MASA of $SO(6)$ generated by the Dirac matrices.

The right-handed toroidal MASA in H_{288} corresponds to the toroidal Cartan subgroup of $SU(3)_{\text{eff}}$.

Thus the finite group H_{288} is the discrete shadow of the continuous MASA structure of $SU(4)$.

10.3 Twistor interpretation

The compact twistor space is $\mathbb{C}^4$ with $SU(4)$ acting linearly. The split twistor space is $\mathbb{R}^4$ with $SL(4,\mathbb{R})$ acting linearly.

Wilson's finite group H_{288} acts on $\mathbb{R}^4$ as a discrete twistor symmetry, providing the finite analogue of the continuous twistor geometry of $SU(4)$.

Conclusion. The finite group H_{288} is the discrete MASA shadow of the continuous MASA structure of $SU(4)$, and hence of the compact twistor branch of $SL(4,\mathbb{C})$.

11 Relation to Euclidean Quantum Field Theory

Euclidean quantum field theory provides a dynamical mechanism for the emergence of Lorentz symmetry, the corrected Dirac equation, and the electroweak structure. The key axioms are:

- Euclidean action,
- Laplace-transform propagators,

- perfect-fluid vacuum,
- vacuum-exchange equilibrium,
- effective metric from vacuum polarisation,
- emergent Lorentz symmetry.

11.1 Emergent Lorentz symmetry

The effective metric is defined by the second derivative of the vacuum polarisation function:

$$g_{\mu\nu}^{\text{eff}} \propto \left.\frac{\partial^2 \Pi(p)}{\partial p_\mu \partial p_\nu}\right|_{p=0}.$$

This metric has Lorentzian signature $(-,+,+,+)$, even though the underlying spacetime is Euclidean.

Thus Lorentz symmetry is not fundamental but emergent.

11.2 Discrete electroweak symmetry

The electroweak group

$$SU(2)_L \times U(1)_{\text{EM}}$$

arises from vacuum polarisation of the $SU(2)_R$ gauge fields, which are suppressed by infinite magnetic self-inductance.

The left-handed $SU(2)_L$ remains non-abelian.

The right-handed $SU(2)_R$ collapses to $U(1)_{\text{EM}}$.

This continuous asymmetry corresponds precisely to the discrete asymmetry in H_{288}:

$$SU(2)_L: \text{ non-toroidal MASA}, \qquad SU(2)_R: \text{ toroidal MASA}.$$

11.3 Discrete vacuum geometry

The finite group H_{288} encodes the discrete vacuum geometry of the electroweak sector. Its non-toroidal MASA on the left corresponds to the non-toroidal vacuum polarisation of the weak interaction. Its toroidal MASA on the right corresponds to the toroidal vacuum polarisation of the emergent electromagnetic interaction.

Conclusion. Wilson's finite group H_{288} is the discrete analogue of the continuous vacuum-polarisation geometry of Euclidean quantum field theory.

12 Relation to the Corrected Chiral Dirac Equation

The corrected Dirac equation arises from vacuum polarisation of the $SU(2)$ gauge fields:

$$(i\gamma^\mu \partial_\mu - M_{\text{vac}}(x))\psi(x) = 0,$$

where $M_{\text{vac}}(x)$ is the vacuum-polarisation mass operator.

12.1 Chiral structure

The Lorentz algebra satisfies

$$\mathfrak{so}(3,1)_{\mathbb{C}} \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

corresponding to left- and right-handed Weyl spinors.

The gamma matrices are built from two copies of $SU(2)$:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

Thus the Dirac algebra contains two $SU(2)$ factors, matching the $SU(2)_L \times SU(2)_R$ structure of $SO(4)$.

12.2 Discrete chiral asymmetry

The corrected Dirac equation requires:

- strong vacuum coupling for left-handed fermions,

- weak or absent vacuum coupling for right-handed fermions.

This continuous chiral asymmetry corresponds precisely to the discrete asymmetry in H_{288}:

- $SU(2)_L$ contains a non-toroidal MASA,
- $SU(2)_R$ contains a toroidal MASA.

12.3 Mass hierarchy

The flavour phase $\mathbb{Z}_3$ in H_{288} corresponds to the directionality of the vacuum-polarisation mass operator. The flavour permutations S_3 correspond to the discrete analogue of CKM mixing.

Conclusion. Wilson's finite group H_{288} encodes, in discrete form, the chiral structure and mass hierarchy of the corrected Dirac equation.

13 Discrete Twistor Geometry

The twistor group $SL(4,\mathbb{C})$ admits three real forms:

$$SU(4), \qquad SL(4,\mathbb{R}), \qquad SU(2,2),$$

each giving rise to a distinct twistor geometry.

- $SU(2,2)$ is the conformal twistor group of Minkowski space.
- $SU(4)$ is the compact twistor group associated with vacuum polarisation.
- $SL(4,\mathbb{R})$ is the split twistor group associated with Wilson's electroweak real form.

In the continuous theory, twistor space is $\mathbb{C}^4$ or $\mathbb{R}^4$ with the real form acting linearly. In the discrete theory, Wilson's finite group H_{288} acts on $\mathbb{R}^4$ as a discrete twistor symmetry.

13.1 Discrete fibres

The continuous twistor fibration

$$\mathbb{CP}^3 \longrightarrow \mathrm{Gr}(2,4,\mathbb{C})$$

has fibres corresponding to complex lines in $\mathbb{C}^4$.

In the split real form, the analogous fibration

$$\mathbb{RP}^3 \longrightarrow \mathrm{Gr}(2,4,\mathbb{R})$$

has fibres corresponding to real lines in $\mathbb{R}^4$.

Wilson's finite group H_{288} acts discretely on these fibres, providing a finite analogue of the continuous twistor geometry.

13.2 MASAs and twistor lines

In the continuous theory, twistor lines correspond to maximal abelian subgroups of $SL(4,\mathbb{C})$.

In the discrete theory, twistor lines correspond to maximal abelian subgroups of H_{288}.

The non-toroidal MASA in H_{288} corresponds to the non-toroidal MASA of $SO(6)$ generated by the Dirac matrices. The toroidal MASA in H_{288} corresponds to the toroidal Cartan subgroup of $SU(3)_{\text{eff}}$.

13.3 Discrete twistor fibre

Thus H_{288} is the discrete twistor fibre of the split real form $SL(4,\mathbb{R})$, in precise analogy with:

- the continuous twistor fibre of $SU(2,2)$,
- the continuous twistor fibre of $SU(4)$.

Conclusion. Wilson's finite group H_{288} provides a discrete twistor geometry for the $SL(4,\mathbb{R})$ branch of $SL(4,\mathbb{C})$, encoding the MASA structure required by electroweak and flavour physics.

14 Physical Interpretation

Wilson's finite group H_{288} encodes, in discrete form, the essential features of electroweak and flavour physics.

14.1 Chirality

The left-handed $SU(2)_L$ contains a non-toroidal MASA. This encodes the non-abelian, parity-violating nature of the weak interaction.

The right-handed $SU(2)_R$ contains a toroidal MASA. This encodes the abelian nature of the emergent electromagnetic interaction.

14.2 Parity violation

The asymmetry between the non-toroidal MASA on the left and the toroidal MASA on the right encodes parity violation:

$$SU(2)_L \neq SU(2)_R.$$

14.3 Flavour structure

The quotient

$$S_3 \times \mathbb{Z}_3$$

encodes:

- S_3: flavour permutations (discrete CKM-like mixing),
- $\mathbb{Z}_3$: flavour phases (mass hierarchy directionality).

14.4 Mass hierarchy

The $\mathbb{Z}_3$ phase distinguishes directions in flavour space, providing a discrete analogue of the vacuum-polarisation mass operator.

14.5 Emergent colour

The toroidal MASA in $SU(2)_R$ corresponds to the toroidal colour symmetry of $SU(3)_{\text{eff}}$ arising from the compactification

$$SL(3,\mathbb{R}) \to SU(3)_{\text{eff}}.$$

14.6 Twistor interpretation

The action of H_{288} on $\mathbb{R}^4$ provides a discrete twistor symmetry for the split real form $SL(4,\mathbb{R})$, complementing the continuous twistor symmetries of $SU(4)$ and $SU(2,2)$.

Conclusion. Wilson's finite group H_{288} encodes chirality, parity violation, flavour structure, mass hierarchy, emergent colour, and discrete twistor geometry in a single unified finite-group framework.

15 Conclusion

Wilson's finite-group quantisation of the split real form $SL(4,\mathbb{R})$ provides a mathematically precise and physically compelling discrete analogue of twistor geometry.

The group

$$H_{288} = (G_L \times 2T_R)/\mathbb{Z}_2$$

is the unique finite subgroup of $SU(2)_L \times SU(2)_R$ satisfying:

- the presence of Q_8 and $\mathbb{Z}_3$,
- chirality,
- parity violation,
- flavour directionality,
- mass hierarchy,
- and clean embedding into $SO(4) \subset SL(4,\mathbb{R})$.

Its structure

$$Q_8^{(L)} \times Q_8^{(R)} \triangleleft H_{288} \qquad \text{with quotient} \qquad S_3 \times \mathbb{Z}_3$$

encodes the discrete geometry of spin, flavour, and colour.

Its MASA structure encodes the toroidal and non-toroidal symmetries required by electroweak physics.

Its action on $\mathbb{R}^4$ provides a discrete twistor fibre for the split real form $SL(4,\mathbb{R})$, complementing the continuous twistor fibres of $SU(4)$ and $SU(2,2)$.

Final statement. Wilson's finite group H_{288} is the discrete MASA shadow of the $SL(4,\mathbb{R})$ electroweak real form, uniquely encoding the non-toroidal flavour geometry and toroidal colour geometry required by twistor-based unification.

References

Wilson, R. A, Quantisation of SL(4,R), August 2, 2026

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